

The quartic oscillator in an external field and the statistical physics of highly anisotropic solids

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Abstract

The statistical mechanics of 1D and 2D Ginzburg-Landau systems is evaluated analytically, via the transfer matrix method, using an expression of the ground state energy of the quartic anharmonic oscillator in an external field. In the 2D case, the critical temperature of the order/disorder phase transition is expressed as a Lambert function of the inverse inter-chain coupling constant.

1 Introduction

The physics of 1D Ginzburg-Landau systems, described by a polynomial Hamiltonian, with quartic anharmonicities, is of utmost interest, for several domains, from quantum field theory to structural phase transitions. 2- or 3D aggregates of such chains describe anisotropic anharmonic solids. Examples of this kind of 2D systems are the ultra-thin ferroelectric films, obtained quite recently [1]. In ultra-thin polymeric ferroelectric films, the strong dipoles constituting the polymer chain allow the manipulation of the individual monomer by a local field, making this class of materials very attractive for molecular electronics [2][Cai2004].

Theoretically, one of the most popular approaches to the study of Ginzburg-Landau anisotropic systems is the transfer matrix method. With this method, the evaluation of the free energy of a $(N + 1)$ - dimensional classical system is equivalent to the evaluation of the ground state energy of a quantum N - dimensional system, described by an neffective Schrodinger equation. The classical $(N + 1)$ - dimensional system and its quantum N - dimensional counterpart are sometimes termed dual systems [3]. Any progress in understanding the physics of such a system is mirrored by a progress in understanding its dual system. In traditional approaches [4], [5], the solution of the quantum problem is used in order to find its classical counterpart. However, in some Monte Carlo simulations [6], the classical 2D system was studied in order to solve the quantum 1D system.

Technically speaking, the transfer matrix approach, as developed by Scalapino and his co-workers [4], [7] is mainly a convenient method of evaluating the partition sum, expressed as a functional integral. The first decade of investigations of this subject - until the early '80s - was dominated by significant analytical results. For the next two decades, the efforts were mainly focused on numerical investigations. However, in the recent years, some exact results, or at least analytical approximations, obtained in the study of the quartic oscillator, show a change of this tendency. They allow us to give analytical responses to some important problems, which have been received, previously, only numerical solutions.

To this trend belongs also the present paper. We use an analytical approximation of the ground state energy of a quartic oscillator in an external field, obtained by Van der Straeten and Naudts [8], in order to find the exact statistical mechanics of a chain of classical anharmonic oscillators, and the mean-field approximation of the statistical mechanics of a planar array of coupled chains.

In fact, the formula proposed in [8] for the ground state energy of a quartic oscillator in an electric field is merely an ansatz than a rigorously obtained result. Also, it depends only on the external field - any dependence of temperature of the effective mass of the oscillator, essential for the description of the statistical mechanics of the linear chain, is lost, being replaced by some numerical parameters. In our approach, both these drawbacks of the Van der Straeten-Naudts solution are removed. The temperature dependence is restored using a comparison with similar results obtained for the 1D Ising model in external field. The adequacy of the ansatz is checked and confirmed by the physical character of its predictions and, more than this, by the fact that it gives results obtained using alternative methods. This is an interesting example of the fact that, while studying dual systems, a progress made in understanding one of them is mirrored in the understanding of the other.

The outline of this paper is as follows. In section 2, we shall sketch the general frame of the problem, at least for sake of clearly defining the notations. In Section 3, we shall expose in some details the Van der Straeten-Naudts solution, in order to properly understand its advantages and drawbacks. In Section 4, this solution, with restored temperature dependence, is used in order to find the electrical polarization of the anharmonic chain. Section 5 is devoted to the physics of 2- and 3D arrays of such chains, treated in the mean-field approximation. A new analytic formula, for the critical temperature of the order/disorder transition, as a function of the inter-chain coupling, is obtained. Its small coupling limit coincides with a formula obtained in [10], confirming the adequacy of the ansatz contained in the Van der Straeten-Naudts solution. The last section is devoted to conclusions.

2 Chains of anharmonic oscillators

Basically, the physical system to be studied in this paper is a chain of classical anharmonic oscillators, described by the Hamiltonian:

$$H_{cl} = \sum_{i=1}^N \frac{1}{2} m_i \left(\frac{du_i}{dt} \right)^2 + \sum_{i=1}^N \left[\left(\frac{1}{2} A u_i^2 + \frac{1}{4} B u_i^4 \right) + \sum_{j=1}^N \frac{1}{2} c_{ij} (u_i - u_j)^2 \right] \quad (1)$$

Each oscillator is an atom; i, j indicate lattice sites (equilibrium positions of atoms); u_i - displacements of displacing atoms with respect to some heavy ions or reference lattice. Periodic boundary conditions are assumed. The coefficient A is defined by the attractive interactions of the mobile atom with the reference lattice, B - by short-range repulsive interactions, and c_{ij} - by elastic interactions between displacing atoms. If (1) describes a lattice which is unstable against a displacive transition, $A < 0$, $B > 0$, $c_{ij} > 0$. In this situation, the potential energy on site appearing in (1) has two minima, at

$$u_0 = \left(\frac{|A|}{B} \right)^{1/2} \quad (2)$$

We shall replace the discrete Hamiltonian (1) with a continuum representation:

$$H_{cl} = \int \frac{dx}{l} \left[\frac{1}{2m} p(x)^2 + \frac{A}{2} u(x)^2 + \frac{B}{4} u(x)^4 + \frac{1}{2} m c_0^2 \left(\frac{du}{dx} \right)^2 \right] \quad (3)$$

where l is the lattice spacing, c_0 - the sound velocity, and $x_j = jl$ locates an atom in the continuum representation.

The equilibrium thermodynamics of this 1D model can be obtained from the classical partition function:

$$Z = \int \mathcal{D}u \mathcal{D}p e^{-\beta \mathcal{H}(u,p)} \quad (4)$$

where, as usual,

$$Z_p = (2\pi k_B T)^{N/2} \quad (5)$$

and

$$Z_u = e^{-N\beta\epsilon_0} \quad (6)$$

with ϵ_0 - the ground state energy of the Schrodinger equation:

$$\left(-\frac{1}{2m^*} \frac{d^2}{du^2} + \frac{A}{2} u^2 + \frac{B}{4} u^4 \right) \Psi_n(u) = \epsilon_n \Psi_n(u) \quad (7)$$

$$m^* = m \frac{c_0^2}{l^2 k_B^2 T^2} \quad (8)$$

The free energy F of the chain is:

$$F = -k_B T \ln Z_p Z_u = F_p + F_u \quad (9)$$

but, in fact, only the "interaction term", F_u , produces interesting physical effects. The free energy per particle is:

$$f_u = \epsilon_0 \quad (10)$$

In the presence of an electric field, a term

$$-\mathcal{E} q u_i = -p u_i$$

is added to the on-site energy (the notation used in the r.h.s. intends to compromise with that used in [8]), and the quantum Hamiltonian (7) becomes:

$$H_q = -\frac{1}{2m^*} \frac{d^2}{du^2} + \frac{A}{2} u^2 + \frac{B}{4} u^4 - p u \quad (11)$$

So, the average value of the displacement can be written as:

$$\langle u \rangle = -\frac{\partial \epsilon_0(p)}{\partial p} \quad (12)$$

where $\epsilon_0(p)$ is the ground state energy of H_q . Also, using any of the Hamiltonians (7), (11), we can compute the average value of other powers of the order parameter:

$$\langle u^2 \rangle = -2 \frac{\partial \epsilon_0}{\partial |A|}, \quad \langle u^4 \rangle = 4 \frac{\partial \epsilon_0}{\partial B}$$

The physics of such a system is quite simple; we shall discuss it starting with the discrete variant, described by the Hamiltonian (1). In the absence of an external field, each atom oscillates near one of the two minima, given by (2); the average value of the displacement is zero, at any temperature. For a planar array of such chains - so, for a 2D system - the situation is quite different. The inter-chain interaction limits the effect of fluctuations, and the average displacement along a chain, $\langle u \rangle$, can have a non-zero value, if the temperature is lower than a critical value, T_c . So, the system displays an order/disorder phase transition, similar to a magnetic/paramagnetic transition, in a magnetic system. Referring again to the 1D system, an external electric field can produce a state with $\langle u \rangle \neq 0$. Also, the effect of inter-chain interactions, in a 2D system, can be treated, in the mean-field approximation, as an effective external field, able to trigger a phase transition. For the description of such a phase transition, it is important to evaluate the critical temperature as a function of microscopic quantities, mainly of inter-chain coupling. Also, quantities like the order parameter $\langle u \rangle$,

the polarization and the electric susceptibility play a similar role to magnetic moment, magnetization and magnetic susceptibility. In fact, there is an evident connection between the atomic chain with deep on-site wells and the 1D Ising model, at least for small external fields.

In principle, the formulae given in this section, mainly (10), (12), can provide a quite complete description of the "exact" statistical mechanics of the 1D system (if the eigenvalue ϵ_0 is known "exactly"), and a mean-field variant of such a description, for the 2D system. However, as it is wellknown, the eigenvalues of (11) cannot be evaluated exactly, even for $p = 0$. Recently, an approximate analytical expression for $\epsilon_0(p)$ has been obtained in [8]. We shall take advantage of this formula in order to find the polarization, the electrical susceptibility and other properties of the chain of classical anharmonic oscillators, and of the 2D aggregates of such chains. The outputs of our calculations will be also a test of the adequacy of the expression of $\epsilon_0(p)$ given by [8].

In the next section, we shall briefly expose how the expression of $\epsilon_0(p)$ has been obtained, in order to properly understand its validity and its limitations.

3 The quantum double well anharmonic oscillator in an external field

We shall briefly describe the results obtained recently by Van der Straeten and Naudts [8] on this subject. The Hamiltonian:

$$H = \frac{P^2}{2m} + \sum_{i=0}^I \lambda_i Q^i \quad (13)$$

is written in terms of annihilation and creation operators a, a^+ of the ordinary harmonic oscillator with mass m and frequency ω_0 . Defining

$$r^2 = \frac{\hbar}{m\omega_0} \quad (14)$$

and using a formula for normal ordering derived in [11], the Hamiltonian (1) takes the form:

$$H = -\frac{\hbar^2}{4mr^2} (a - a^+)^2 + \sum_{i=0}^I \lambda_i \left(\frac{r}{\sqrt{2}} \right)^i \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \frac{i!}{2^k k!} \sum_{j=0}^{i-2k} \frac{(a^+)^{i-2k-j} a^j}{j! (i-2k-j)!} \quad (15)$$

The expectation value of (15) in a certain state $|\psi_t\rangle$, depending on a parameter t , will be minimized

$$\frac{\partial}{\partial r^2} \langle \psi_t | H | \psi_t \rangle = 0 \quad (16)$$

in order to obtain an equation which can determine the parameter t .

For a quartic oscillator in an external field, $\lambda_1 = -p$, $\lambda_2 = \frac{\alpha}{2}$, $\lambda_3 = 0$, $\lambda_4 = \frac{\beta}{4}$, $I = 4$, and (16) becomes:

$$(2t+1) \left(2\lambda_2 r_0^4 - \frac{\hbar^2}{m} \right) + 6\lambda_4 r_0^6 (2t^2 + 2t + 1) = 0 \quad (17)$$

If t is known - or, equivalently, if the choice of the state vector $|\psi_t\rangle$ has been done - eq. (17) will give the value of r_0 ; this cubic equation has indeed one real root, if the potential has two wells ($\alpha < 0$). The choice of t has no clear physical significance. The authors adopt the variant $t = N/2$, where N is the value at which the $|\psi_t\rangle$ basis is truncated, for reasons of rapid convergence of numerical calculations. ω is also fixed numerically. The choice of a shallow well, with $\alpha = 2\lambda_2 = -2$, $\beta = 4\lambda_4 = 1$, transforms (17) in an equation with numerical coefficients, which gives a numerical value for r_0 .

Finally, the ground state eigenenergy in the presence of an electric field, $\epsilon_0(p)$, is, for small and moderate values of the field:

$$\epsilon_0(p) \simeq \epsilon_0(0) - |a| p \tanh \omega p \quad (18)$$

where the coefficients a , ω are determined numerically. Their "exact" value depends on the truncation of basis functions $|\psi_t\rangle$. In fact, this formula of $\epsilon_0(p)$ is not rigorously deduced, but is merely an interpolation between the "small" and "moderate" field cases. The corectness of this ansatz could be evaluated according to its success of giving reasonable predictions for the statistical physics of the Ginzburg-Landau systems, through the transfer matrix method.

For large fields,

$$\epsilon_0(p) \simeq A_0 + B_0 p^{4/3} \quad (19)$$

with A_0 , B_0 - numerical constants. A similar - but more general - result has been obtained by Bronzan and Sugar [12], for the energy levels of a potential $V(x) = A_1 x + x^4$. It is clear that, for large fields (with Bronzan and Sugar's notations, for $A_1 > 15$), the effect of the quadratic (harmonic) term of the potential is irrelevant, at least for the leading terms.

Let us define what we mean here by a "large" field. In the absence of an external field, the particle oscillates in a symmetric, double-well potential:

$$V_s(x) = \frac{\alpha}{2} x^2 + \frac{\beta}{4} x^4 \quad (20)$$

In the presence of an electric field, one gets a total, asymmetric, potential:

$$V_{as}(x) = \frac{\alpha}{2} x^2 + \frac{\beta}{4} x^4 - px \quad (21)$$

The "electric" term favors one of the two wells; this one becomes deeper, while the other one - more shallow, disappearing when the electric field is larger than a certain critical value p_c . It is easy to find p_c by analyzing the extremum points of $V_{as}(x)$, which are the roots of the equation:

$$\frac{dV_{as}(x)}{dx} = \beta x^3 + \alpha x - p = 0 \quad (22)$$

where $\alpha < 0$, $\beta > 0$, $p > 0$. This equation may have one or three real roots, according to the sign of the discriminant, proportional to the expression:

$$-\frac{|\alpha|^3}{27\beta} + \frac{p^4}{4}$$

which changes its sign at the critical value:

$$p_c = \left(\frac{4}{27} \frac{|\alpha|^3}{\beta} \right)^{1/4} \quad (23)$$

If we presume a Landau-type temperature dependence of the parameter α , i.e. $\alpha = \alpha' (T - T_c^{MF})$, then the critical field has also a T -dependence, of the form:

$$p_c = p_{c0} |T - T_c^{MF}|^{3/4} \quad (24)$$

where we have used the standard notations of the Landau theory of phase transitions (see for instance [13]).

4 The electrical polarization of the anharmonic chain

The polarization is proportional to the average displacement of atoms along the chain; according to (12), one obtains:

$$\langle Q \rangle = \tanh \omega p + \frac{\omega p}{\cosh^2 \omega p}, \quad p < p_c \quad (25)$$

For small fields,

$$\langle Q \rangle = \omega p (1 + \dots) \quad (26)$$

and, comparing with the magnetization of the Ising model for small magnetic fields [14],

$$M = \beta H e^{2\beta E_1} + \mathcal{O}(\beta^2 H^2)$$

we can identify the coefficient ω as

$$\omega \simeq \beta e^{2\beta c u_0^2} \quad (27)$$

Qualitatively, (25) has a correct behaviour, in the sense that it is a monotonically increasing function of p , which saturates asymptotically. The same behaviour occurs in the 1D Ising model in an external field. Also, the comparison with the Ising model recovers the temperature dependence of the numerical constant

ω , entering in the solutions obtained in [8]. This recovery is essential in order to obtain the statistical mechanics of the systems under scrutiny.

With (25), (27), we find for the susceptibility of the chain the following formula:

$$\chi = \frac{2\omega}{\cosh^2 \omega p} \left(1 - \frac{\omega p \tanh \omega p}{\cosh \omega p} \right), \quad p < p_c \quad (28)$$

5 The spatial array of Ginzburg-Landau chains

Let us consider a planar array of N chains. The n -th chain ($1 \leq n \leq N$) is described by a Hamiltonian $H_{cl}^{(n)}$, where $H_{cl}^{(n)}$ is obtained from H_{cl} , eq.(3), replacing $u(x) \rightarrow u_n(x)$, $p(x) \rightarrow p_n(x)$. Cyclic boundary conditions are imposed, so that the chains $n = 1$ and $n = N+1$ coincide. The interaction between chains n, n' is given by:

$$H_{int} = \frac{1}{2} \sum_{n \neq n'} D_{nn'} [u_n(x) - u_{n'}(x)]^2 \quad (29)$$

In a mean field approach, the Hamiltonian (29) is replaced by:

$$-2q_i D \langle u \rangle \sum_n u_n(x), \quad i = 2, 3 \quad (30)$$

where q_i is the number of near neighbors, in i dimensions; we shall focus here, however, only on the 2D case. The evaluation of the statistical physics of the 2D system reduces, through the matrix transfer method, to the study of a Schrodinger equation describing a quartic oscillator in an external field $\Gamma \langle u \rangle$ [3]:

$$\left[-\frac{1}{2m^*} \frac{d^2}{du^2} - \frac{1}{2} |A| u^2 + \frac{1}{4} B u^4 - \Gamma \langle u \rangle u \right] \psi_n(u) = \epsilon_n \psi_n(u) \quad (31)$$

Measuring the energy in units $|V_0| = A^2/4B$ and making the changes:

$$u \rightarrow \left(\frac{2|A|}{B} \right)^{1/2} u; \quad (m^*)^{-1/2} \rightarrow \left(\frac{|A|}{B} \right) \left(\frac{1}{2} |A| \right) \mu; \quad \Gamma \rightarrow \frac{1}{8} |A| \gamma \quad (32)$$

we get the following eigenvalue problem:

$$\left[-\frac{1}{2} \mu \frac{d^2}{du^2} - 4u^2 + 4u^4 - \gamma \langle u \rangle u \right] \psi_n(u) = \epsilon_n \psi_n(u) \quad (33)$$

where:

$$u_0 = 2^{-1/2}, \quad \mu = \frac{1}{4} \left(\frac{k_B T}{V_0} \right) \left(\frac{|A|}{C} \right)^{1/2}; \quad \mu \sim T \quad (34)$$

In the classical paper of Bishop and Krumhansl [BK], the eigenvalue problem is solved numerically. Our contribution, to this point of the problem, is the following: using the analytical solution (anzatz) of Van der Straeten and Naudts [8], we shall obtain analytical formulae for the critical temperature of the 2D transition, and the critical behaviour of the planar array of chains.

Let us now outline our approach to the 2- and 3D problems and describe the main results obtained.

The mean field equation

Replacing in (11)

$$p = \gamma \langle u \rangle \quad (35)$$

we get the self-consistent equation:

$$bx = x + \tanh x - x \tanh^2 x, \quad b = \frac{1}{\omega\gamma}, \quad x = \omega\gamma \langle u \rangle \quad (36)$$

For $b < 2$, the line described by the left hand side of the equation has a non-zero intersection with the curve described by the right hand side, consequently a value $\langle u \rangle \neq 0$ does exist; so, an ordered state appears, meaning that the 2D system displays a phase transition. Consequently, $b = 2$ represents the critical condition, giving the expression of the transition temperature.

Evaluation of the critical temperature T_c

According to eq. (27),

$$\omega = \frac{1}{k_B T} \exp \left(\frac{2}{k_B T} c u_0^2 \right) \quad (37)$$

and the critical condition can be written as:

$$\omega = \frac{1}{k_B T} \exp \left(\frac{2}{k_B T} c u_0^2 \right) \quad (38)$$

Putting

$$\frac{2}{k_B T_c} c u_0^2 = \xi \quad (39)$$

it takes the form of the transcendental Lambert-Euler equation:

$$\xi e^\xi = \frac{c u_0^2}{\gamma} \quad (40)$$

Its solution is given by the Lambert function W [9] :

$$\xi = W \left(\frac{c u_0^2}{\gamma} \right) \quad (41)$$

The Lambert function has the following behaviour, for small, respectively large value of its argument:

$$W(\xi) \sim \begin{cases} \xi, & \xi \sim 0 \\ \ln \xi, & \xi \rightarrow \infty \end{cases} \quad (42)$$

So, for small values of the inter-chain coupling, proportional to γ ,

$$T_c \sim \frac{1}{|\ln \gamma|} \quad (43)$$

This result coincides to that obtained by Scalapino, Imry and Pincus [10], using a different approach. For moderate values of the coupling,

$$T_c \sim \gamma \quad (44)$$

The formulae (43), (44) confirm, qualitatively, the behaviour of T_c , obtained numerically in [15]. Actually, it corrects their interpretation for small γ (considered by these authors to be exponential), and provides an analytical expression for T_c , valid for any value of the inter-chain coupling.

The critical behaviour of the order parameter

In order to obtain the critical behaviour of the order parameter, we have to examine the self-consistent condition (36) for temperatures close to T_c , where, putting $\omega(T_c) = \omega_c$,

$$\omega = \omega_c + \Delta\omega, \quad \Delta\omega \ll \omega_c \quad (45)$$

So,

$$\frac{1}{\omega\gamma} = 2 - 2\gamma\Delta\omega \quad (46)$$

Expanding the hyperbolic functions near the origin, we get:

$$\langle u \rangle \sim |T - T_c|^{1/2} \quad (47)$$

showing a mean field critical behaviour, as we can expect, taking into account the methodology itself. However, our approach is interesting, due to the fact that it also allows a simple, analytical determination of the coefficient of $|T - T_c|^{1/2}$.

The expression of the electrical susceptibility

In the presence of an external electric field, the Hamiltonian obtained through the transfer matrix approach is:

$$H_q^{(\text{modif})} = -\frac{1}{2m^*} \frac{d^2}{du^2} + \frac{1}{2}Au^2 + \frac{1}{4}Bu^4 - \Pi u, \quad \Pi = \mathcal{E} + \gamma \langle u \rangle \quad (48)$$

So, the ground state energy becomes:

$$\epsilon_0(\Pi) = \epsilon_0(0) - |a| \Pi \tanh \omega \Pi \quad (49)$$

and the self-consistency condition takes the form:

$$\langle u \rangle = \tanh(\mathcal{E} + \gamma \langle u \rangle) + \omega(\mathcal{E} + \gamma \langle u \rangle) [1 - \tanh^2 \omega(\mathcal{E} + \gamma \langle u \rangle)] \quad (50)$$

which can be also written as:

$$(b-1)y - e_0 = \tanh y - y \tanh^2 y, \quad y = \omega(\mathcal{E} + \gamma \langle u \rangle), \quad e_0 = -\frac{3\mathcal{E}}{4\Gamma} \quad (51)$$

On this equation, one can easily see that the line defined by the left hand side has always an intersection with the curve defined by the right side, in a point having the abscise $x > 0$. Consequently, in the presence of an external field, $\langle u \rangle \neq 0$, as expected. For small values of y , the self-consistency condition gives the following equation:

$$\frac{4}{3}y^3 + (b-2)y - e_0 = 0 \quad (52)$$

The physically interesting regime is still that corresponding to $b < 2$, so, putting

$$\frac{\Delta T}{T_c} = \tau$$

we get the solution:

$$\langle u \rangle = \frac{1}{\omega\Gamma} \left(\frac{3e_0}{8} \right)^{1/3} \left\{ \left[1 + \left(1 - \frac{8}{9e_0^2} \tau^3 \right)^{1/2} \right]^{1/3} + \left[11 \left(1 - \frac{8}{9e_0^2} \tau^3 \right)^{1/2} \right]^{1/3} \right\} - e_0 \quad (53)$$

The exponent of the critical isotherm can be obtained immediatly, putting $\tau = 0$ in the previous equation:

$$\langle u \rangle = \frac{1}{\omega\Gamma} \left(\frac{3\mathcal{E}}{8\Gamma} \right)^{1/3} + \dots, \quad \langle u \rangle^3 \sim \mathcal{E} \quad (54)$$

So, the exponent of the critical isotherm is $\delta = 3$, as expected from the general theory [Stanley].

However, the critical behaviour of the susceptibility can be obtained simpler, through the differentiation of (52). Keeping only the smallest terms, we have:

$$\frac{dy}{de_0} = \frac{1}{b-2} \sim \frac{1}{\Delta T} \quad (55)$$

As this report is proportional to the electrical susceptibility, we find for its critical behaviour a Curie-type law:

$$\chi(T \simeq T_c) \sim \frac{1}{T - T_c} \quad (56)$$

The coefficient of the $(T - T_c)^{-1}$ factor can be easily obtained, using standard methods (see for instance [16], Ch.6).

6 Conclusions

In this paper, we have applied the transfer matrix method in order to study the statistical mechanics of 1D and 2D Ginzburg-Landau systems - chains of anharmonic oscillators, or planar arrays of such chains, which can simulate ultra-thin films of ferroelectric systems. The starting point of our approach is a partially numeric / partially analytic expression for the ground state energy of a quartic oscillator in an electric field, $\epsilon_0(p)$, recently obtained by Van der Straeten and Naudts. In fact, the formula for $\epsilon_0(p)$ is mainly an ansatz than a rigorous result. Comparing the predictions of this formula, via the transfer matrix method, for the polarization of the Ginzburg-Landau chain in an electric field, with the small-field limit of the exact formula of the magnetization of the 1D Ising model in a magnetic field, the temperature dependence of the ground state energy of the quartic oscillator in an external field is restored. Due to this fact, the expression of the ground state energy $\epsilon_0(p)$ can be used, through a mean field approximation, in order to find the statistical mechanics of the 2D Ginzburg-Landau system. The method gives an analytical formula of the critical temperature of the 2D order/disorder transition, expressed as a Lambert function of the inverse inter-chain coupling.

Through this approach, we are able not only to find the statistical mechanics of 1D and 2D Ginzburg-Landau systems, but also to confirm the adequacy of the ansatz made by Van der Straeten and Naudts while proposing a formula for the ground state energy of a quartic oscillator in external field.

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